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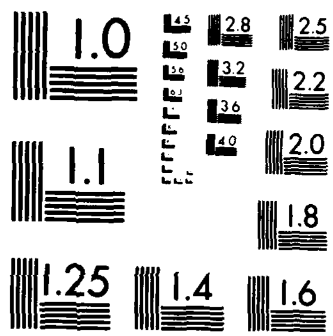
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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Uniform stochastic orderings of random variables are expressed as total positivity of density, survival and distribution functions. The orderings are called uniform because each is a stochastic order that persists under conditioning to a

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20. ABSTRACT CONTINUED:

family of intervals - for example, the family consisting of all intervals of the form $(-\infty, x]$. This paper is concerned with the preservation of uniform stochastic ordering under convolution, mixing and the formation of coherent systems. A general TP_2 result involving preservation of total positivity under integration is presented and applied to convolutions and mixtures of distribution and survival functions. Logconcavity of distribution, survival and density functions characterises random variables that preserve the various orderings under addition. Likewise, random variables that preserve orderings under mixing are characterized by TP_2 distribution and survival functions.

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TP_2 Orderings

by

James Lynch[1], Gillian Mimmack[2], and Frank Proschan[2]

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The Florida State University
Department of Statistics
Tallahassee, Florida 32306

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TP_2 Orderings

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James Lynch[1], Gillian Mimmack[2] and Frank Proschan[2]

Abstract

Uniform stochastic orderings of random variables are expressed as total positivity of density, survival and distribution functions. The orderings are called uniform because each is a stochastic order that persists under conditioning to a family of intervals - for example, the family consisting of all intervals of the form infinity $(-\phi,x]$. This paper is concerned with the preservation of uniform stochastic ordering under convolution, mixing and the formation of coherent systems. A general $TP_2^{\cap}$ result involving preservation of total positivity under integration is presented and applied to convolutions and mixtures of distribution and survival functions. Logconcavity of distribution, survival and density functions characterises random variables that preserve the various orderings under addition. Likewise, random variables that preserve orderings under mixing are characterized by $TP_2^{\cap}$ distribution and survival functions.

to shifts in location or changes in scale. In what follows, we extend these results, considering preservation of uniform ordering under mixing and under the formation of coherent systems and characterizing random variables that preserve uniform stochastic ordering under addition and mixing.

Section 2 contains definitions of the uniform stochastic orderings under consideration. In Section 3 we present a general theorem that applies to convolutions and mixtures in the context of uniform stochastic ordering and is of interest in its own right as a result in the theory of TP_2 functions. In Section 4 we exploit the TP_2 expressions of uniform stochastic orderings to characterize logconcave distribution and survival functions as those that preserve the appropriate uniform stochastic ordering under convolution. In Section 5 we use the general preservation theorem of Section 3 to characterize random variables that will allow uniform stochastic ordering to be transferred from mixing random variables to mixtures. Finally, in Section 6, we consider whether stochastic ordering between corresponding components of identical systems is inherited by the systems. We show, for example, that if the failure rate of one type of component is larger than the failure rate of another type of component, then a k-out-of-n system of independent components has a larger failure rate if components of the first type are used.

2. Definitions and preliminaries.

Let X be a random variable with distribution function given by $F(x) = P(X \le x)$, survival function $\overline{F} = 1 - F$ and, in the absolutely continuous case, density function f.

The uniform stochastic orderings under consideration may be conveniently expressed in terms of TP_2 ordering of functions. Let g_1 and g_2 be an arbitrary pair of functions.

Definition 2.1.

$$g_1 \overset{TP_2}{<} g_2 \iff g_1(x)g_2(y) - g_1(y)g_2(x) \geq 0 \text{ for all } x \leq y.$$

It is evident that $g_1 \overset{TP_2}{<} g_2$ is equivalent to saying that $g_i(x)$ is TP_2 in (i,x) where the domain of g is $A \times B$; in this case $A = \{1,2\}$ and B is some subset of the real line. More generally, we could consider a family of functions $\{g_\alpha, \alpha \in A\}$ where A is some subset of the real line. Then the statement that $g_\alpha(x)$ is TP_2 in (α,x) would indicate that $g_\alpha \overset{TP_2}{<} g_\beta$ for $\alpha < \beta$.

The various uniform stochastic orderings are defined by taking g_1 and g_2 to be distribution, survival and density functions.

Definition 2.2.

(i) $X_{1} \underset{(-)}{\leq} X_2 \iff F_1 \overset{TP_2}{<} F_2 \iff F_1(x)\, F_2(y) - F_1(y)\, F_2(x) \geq 0$ for all $x \leq y$.

(ii) $X_{1} \underset{(+)}{\leq} X_2 \iff \overline{F}_1 \overset{TP_2}{<} \overline{F}_2 \iff \overline{F}_1(x)\, \overline{F}_2(y) - \overline{F}_1(y)\, \overline{F}_2(x) \geq 0$ for all $x \leq y$.

(iii) $X_1 <_\ell X_2 \iff f_1 \overset{TP_2}{<} f_2 \iff f_1(x)\, f_2(y) - f_1(y)\, f_2(x) \geq 0$ for all $x \leq y$.

If $X_{1} \underset{(-)}{\leq} X_2$ we say that X_1 is uniformly smaller than X_2 in the negative direction because $F_1 \overset{TP_2}{<} F_2$ is equivalent to the condition $P(X_1 \leq x | X_1 \leq y) \geq P(X_2 \leq x | X_2 \leq y)$ for all x and y such that $x \leq y$. It is evident that this defines a stochastic ordering of X_1 and X_2 when each is restricted to the same interval of the form $(-\infty, y]$, the ordering of X_1 and X_2 being the same for all such intervals. Similarly, if $X_{1} \underset{(+)}{\leq} X_2$ we say that X_1 is uniformly smaller than X_2 in the positive direction because the condition $\overline{F}_1 \overset{TP_2}{<} \overline{F}_2$ defines a stochastic ordering of X_1 and X_2 when each is restricted to an arbitrarily chosen interval of the form $[x,\infty)$. If $X_1 <_\ell X_2$

we say that X_1 is locally stochastically less than X_2 because this condition is equivalent to requiring that X_1 be stochastically less than X_2 whenever X_1 and X_2 are restricted to the same interval of the form [a,b]. It is easy to prove that the local ordering is the most restrictive in that $X_1 <_{\ell} X_2$ implies $X_{1} \underset{(-)}{\leq} X_2$ and $X_{1} \underset{(+)}{\leq} X_2$, that ordinary stochastic ordering ($X_1 \overset{st}{<} X_2$) is implied by each of the weaker uniform stochastic orderings, and that the relationship between positive and negative uniform stochastic ordering is

$$X_{1} \underset{(-)}{\leq} X_2 \iff -X_{2} \underset{(+)}{\leq} -X_1.$$

Another property we shall use is that positive and negative uniform ordering is preserved under the operation of taking the limit in distribution.

In the case of vectors of random variables, or sets of lifetimes of components in our context, where $\underline{f} = (f_1, \ldots, f_n)$ and $\underline{g} = (g_1, \ldots, g_n)$, we write $\underline{f} \overset{TP_2}{<} \underline{g}$ if $f_i \overset{TP_2}{<} g_i$ for $i = 1, \ldots, n$. By taking $\underline{f}$ and $\underline{g}$ to be vectors of distribution, survival and density functions, we extend the definitions of negative, positive and local uniform stochastic ordering.

In all that follows, we adopt the conventions that "increasing" means "nondecreasing" and "decreasing" means "nonincreasing".

3. A general preservation theorem.

The result below is of general interest because it involves preservation of total positivity under integration. It is therefore given in terms of TP_2 functions although it will be used in later sections concerning convolutions and mixtures.

Theorem 3.1.

Suppose that the distribution function $\overline{H}(\cdot,t)$ is differentiable with respect to t.

(i) $\overline{H}(x,t)$ is TP_2 in (x,t) <=> $\int \overline{H}(x,t)dF_\theta(t)$ is TP_2 in (θ,x) whenever $\overline{F}_\theta(t)$ is TP_2 in (θ,t).

(ii) $H(x,t)$ is TP_2 in (x,t) <=> $\int H(x,t)dF_\theta(t)$ is TP_2 in (θ,x) whenever $F_\theta(t)$ is TP_2 in (θ,t).

To prove Theorem 3.1 the following lemma is needed.

Lemma 3.2.

If $H(x) = 1 - e^{-\lambda x}$ then $\int \overline{H}(x-t)dF_\theta(t)$ is TP_2 in (θ,x) whenever $\overline{F}_\theta(t)$ is TP_2 in (θ,t).

Proof of lemma.

Suppose $\overline{F}_\theta(t)$ is TP_2 in (θ,t). Without loss of generality we assume that θ takes only the values 1 and 2. We need to show that

$$\begin{vmatrix} \int \overline{F}_1(t)h(x-t)dt & \int \overline{F}_2(t)h(x-t)dt \\ \int \overline{F}_1(t)h(y-t)dt & \int \overline{F}_2(t)h(y-t)dt \end{vmatrix} \geq 0 \text{ for } x < y.$$

By the Basic Composition Formula (Karlin (1968), p. 17), the determinant reduces to

$$\iint_{s<t} \begin{vmatrix} \overline{F}_1(s) & \overline{F}_1(t) \\ \overline{F}_2(s) & \overline{F}_2(t) \end{vmatrix} \begin{vmatrix} h(x-s) & h(y-s) \\ h(x-t) & h(y-t) \end{vmatrix} ds\, dt$$

The first determinant above is nonnegative because $\overline{F}_i(t)$ is TP_2 in (i,t) and the second determinant is nonnegative because h is the exponential density function which is logconcave.||

Proof of Theorem.

We prove only (i) as (ii) is proven from (i) since $H(x,t)$ is a distribution function that is TP_2 in (x,t) if and only if $H(-x, -t)$ is a survival function that is TP_2 in (x,t).

=> Let $\overline{H}(x,t)$ be TP_2 in (x,t) and $\overline{F}_\theta(t)$ be TP_2 in (θ,t), where we assume without loss of generality that θ takes only the values 1 and 2. Suppose first that $F_1(F_2)$ has density $f_1(f_2)$. We need to show that

$$D = \begin{vmatrix} \int \overline{H}(x,t) f_1(t)dt & \int \overline{H}(y,t) f_1(t)dt \\ \int \overline{H}(x,t) f_2(t)dt & \int \overline{H}(y,t) f_2(t)dt \end{vmatrix} \geq 0 \text{ for } x < y.$$

Using the Basic Composition Formula (Karlin (1968), p. 17) and integrating the inner integral by parts, we reduce the determinant as follows:

$$D = \iint_{s<t} \begin{vmatrix} \overline{H}(x,s) & \overline{H}(x,t) \\ \overline{H}(y,s) & \overline{H}(y,t) \end{vmatrix} \begin{vmatrix} f_1(s) & f_2(s) \\ f_1(t) & f_2(t) \end{vmatrix} dt\, ds$$

$$= \iint_{s<t} \begin{vmatrix} \overline{H}(x,s) & \partial\overline{H}(x,t)/\partial t \\ \overline{H}(y,s) & \partial\overline{H}(y,t)/\partial t \end{vmatrix} \begin{vmatrix} f_1(s) & f_2(s) \\ \overline{F}_1(t) & \overline{F}_2(t) \end{vmatrix} dt\, ds$$

Since $\overline{H}(x,t)$ is TP_2 in (x,t), we have, for $s<t$ and $x<y$,

$$\frac{\partial\overline{H}(y,t)/\partial t}{\overline{H}(y,t)} \; \frac{\overline{H}(y,t)}{\overline{H}(y,s)} - \frac{\partial\overline{H}(x,t)/\partial t}{\overline{H}(x,t)} \; \frac{\overline{H}(x,t)}{\overline{H}(y,s)} \geq 0.$$

Thus the first determinant in the integral above is nonnegative. Similarly, since $\overline{F}_\theta(t)$ is TP_2 in (θ,t), we have, for $s<t$ and $x<y$,

$$\frac{f_1(s)}{\overline{F}_1(s)} \; \frac{\overline{F}_1(s)}{\overline{F}_1(t)} - \frac{f_2(s)}{\overline{F}_2(s)} \; \frac{\overline{F}_2(s)}{\overline{F}_2(t)} \geq 0.$$

Thus the second determinant in the integral above is also nonnegative.

If one or both of F_1 and F_2 does not have a density, we proceed as follows. Let $\overline{F}_i^\lambda(x) = \int \overline{L}_\lambda(x-t)dF_i(t)$ where $\overline{L}_\lambda(t) = e^{-\lambda t}$. From lemma 3.2 we have that $\overline{F}_i^\lambda(x)$ is TP_2 in (i,x) for every $\lambda>0$. Since F_i^λ is the convolution of an exponential distribution function and F_i, it has a density and therefore the proof above applies. Thus $\int \overline{H}(x,t)dF_i^\lambda(t)$ is TP_2 in (i,x). Then, since F_i^λ converges weakly to

F_i and since $\overline{H}$ is bounded and continuous in t, we have, by the definition of weak convergence,

$$\lim_{\lambda\to\infty} \int \overline{H}(x,t)dF_i^{\lambda}(t) = \int \overline{H}(x,t)dF_i(t) \text{ for each } x.$$

As the TP_2 relation for $\int\overline{H}(x,t)dF_i{}^{\lambda}(t)$ is preserved under this limiting operation, we have that $\int\overline{H}(x,t)dF_i(t)$ is TP_2 in (i,x).

<= Suppose that $\int \overline{H}(x,t)dF_{\theta}(t)$ is TP_2 in (θ,x) whenever $\overline{F}_{\theta}(t)$ is TP_2 in (θ,t). We assume without loss of generality that θ takes only the values 1 and 2. We need to show that $\overline{H}(x,t)$ is TP_2 in (x,t). Let $F_1(F_2)$ denote the distribution function of the random variable that is degenerate at $s_1(s_2)$ where $s_1<s_2$. Since $\overline{F}_i(t)$ is TP_2 in (i,t), we have that $\int \overline{H}(x,t)dF_i(t)$ is TP_2 in (i,x). But $\int \overline{H}(x,t)dF_i(t) = \overline{H}(x,s_i)$ and therefore the previous statement is equivalent to

$$\overline{H}(x,s_1)\ \overline{H}(y,s_2) - \overline{H}(x,s_2)\ \overline{H}(y,s_1) \geq 0 \text{ for } x<y.$$

This is true for any s_1 and s_2 for which $s_1<s_2$.||

4. Preservation of uniform stochastic ordering under convolution.

We suppose that X_1 and X_2 are uniformly stochastically ordered in some way and consider the ordering of $X_1 + Z$ and $X_2 + Z$. We also consider the reverse question, namely, if $X_1 + Z$ and $X_2 + Z$ are uniformly stochastically ordered whenever X_1 and X_2 are, what can be said about Z? Logconcavity enters naturally from the TP_2 characterizations of the various uniform stochastic orderings because, as is well-known, a nonnegative function g is logconcave if and only if $g(x-y)$ is TP_2 in (x,y). Using this fact, and considering scalars θ_1 and θ_2 $(0 \leq \theta_1 \leq \theta_2)$ rather than random variables X_1 and X_2, Keilson and Sumita (1982) show that logconcavity of the distribution, survival and density functions of Z characterizes the random variables that can be respectively negatively, positively and locally ordered according to θ_1 and θ_2 - that is, for example, Z has a logconcave distribution

function if and only if $Z + \theta_1 \underset{(-)}{\leq} Z + \theta_2$ whenever $\theta_1 \leq \theta_2$. Keilson and Sumita (1982) also point out that if Z has a logconcave density function then uniform stochastic ordering of X_1 and X_2 is inherited by X_1+Z and X_2+Z. We extend these results, characterizing logconcave survival and distribution functions as those that preserve under addition not only the order of scalars but also the uniform stochastic order of random variables.

Theorem 4.1.

Suppose that Z is a continuous random variable.

(i) The survival function of Z is $PF_2 \Leftrightarrow X_1+Z \underset{(+)}{\leq} X_2 + Z$ whenever $X_1 \underset{(+)}{\leq} X_2$.

(ii) The distribution function of Z is $PF_2 \Leftrightarrow X_1 + Z \underset{(-)}{\leq} X_2 + Z$ whenever $X_1 \underset{(-)}{\leq} X_2$.

Proof.

Suppose Z has survival function $\bar{H}$ and $X_1(X_2)$ has survival function $\bar{F}_1(\bar{F}_2)$. The condition that $X_1 + Z \underset{(+)}{\leq} X_2 + Z$ whenever $X_1 \underset{(+)}{\leq} X_2$ is equivalent to the condition that $\int \bar{H}(z-t)dF_i(t)$ is TP_2 in (i,z) whenever $\bar{F}_i(t)$ is TP_2 in (i,t). Part (i) therefore follows immediately from Theorem 3.1(i). Part (ii) follows from Theorem 3.1(ii) in a similar fashion.||

5. Preservation of uniform stochastic ordering under mixture.

We consider the question of whether uniform stochastic order between two mixing random variables is inherited by the resultant mixtures and the reverse question, namely, if uniform stochastic order is transferred from mixing random variables to mixtures, what can be said about the variables over which the mixing is done? Keilson and Sumita (1982) consider these questions primarily in the context

of discrete random variables: they show that local stochastic ordering of the mixing random variables is preserved by the mixtures if and only if the variables to be mixed form a finite sequence of locally stochastically ordered random variables. In the same vein, Keilson and Sumita (1982) give conditions under which positive (negative) uniform stochastic ordering is inherited by mixtures of a finite sequence of positively (negatively) uniformly stochastically ordered random variables. We show that the latter results are true in general - that is, positive (negative) uniform stochastic ordering between mixing random variables is inherited by the resultant mixtures if and only if the variables to be mixed form a (not necessarily finite) sequence of positively (negatively) uniformly stochastically ordered random variables.

Theorem 5.1.

Suppose that X_α has distribution function $H(\cdot,\alpha)$ which is differentiable with respect to α, let W_i have distribution function F_i and let Z_i have distribution function $\int \bar{H}(\cdot,\alpha)dF_i(\alpha)$, i=1, 2.

(i) $X_\alpha \underset{(+)}{\leq} X_\beta$ for $\alpha < \beta \Leftrightarrow Z_1 \underset{(+)}{\leq} Z_2$ whenever $W_1 \underset{(+)}{\leq} W_2$.

(ii) $X_\alpha \underset{(-)}{\leq} X_\beta$ for $\alpha < \beta \Leftrightarrow Z_1 \underset{(-)}{\leq} Z_2$ whenever $W_1 \underset{(-)}{\leq} W_2$.

Proof.

Since $X_\alpha \underset{(+)}{\leq} X_\beta$ for $\alpha < \beta$ is equivalent to $\bar{H}(x,\alpha)$ being TP_2 in (x,α), and similarly for $Z_1 \underset{(+)}{\leq} Z_2$ and $W_1 \underset{(+)}{\leq} W_2$, (i) follows immediately from Theorem 3.1 (i).

Similarly, (ii) follows from Theorem 3.1 (ii). ||

6. Preservation of uniform stochastic ordering under the formation of coherent systems.

The uniform stochastic orderings defined above may also be expressed as follows:

$$F_1 \overset{TP_2}{<} F_2 \iff F_1(x)/F_2(x) \text{ is decreasing in } x$$

$$\bar{F}_1 \overset{TP_2}{<} \bar{F}_2 \iff \bar{F}_1(x)/\bar{F}_2(x) \text{ is decreasing in } x$$

$$f_1 \overset{TP_2}{<} f_2 \iff f_1(x)/f_2(x) \text{ is decreasing in } x.$$

In the context of reliability we may view X_1 and X_2 as lifetimes of units: if $F_1 \overset{TP_2}{<} F_2$, we see that a unit that has failed by time t_2 is more likely to have failed by time t_1 $(t_1 < t_2)$ if its lifetime has distribution F_1 rather than F_2. Similarly, if $\bar{F}_1 \overset{TP_2}{<} \bar{F}_2$, a unit of age t_1 with lifetime X_1 is less likely to survive beyond t_2 $(t_1 < t_2)$ than is a unit of the same age (t_1) with lifetime X_2. This is equivalent to the condition that the failure rate of X_1 is greater than or equal to the failure rate of X_2.

The characterizations given above lead to consideration of uniform stochastic ordering of the lifetimes of coherent systems when corresponding components of comparative systems are uniformly stochastically ordered. We show first that if the failure rate of one type of component is higher than that of a second type of component then certain systems have a higher failure rate if components of the first type are used rather than components of the second type.

Theorem 6.1.

Let $h(p)$ be the reliability of a coherent system of n independent and identically distributed components with failure probability p. If $ph'(p)/h(p)$ is decreasing in p then $h(\bar{F}) \overset{TP_2}{<} h(\bar{G})$ whenever $\bar{F} \overset{TP_2}{<} \bar{G}$.

Proof: Suppose first that $F(G)$ is absolutely continuous with failure rate $r(s)$ where $\bar{F} \overset{TP_2}{<} \bar{G}$, or, equivalently, $r \geq s$. Let $R(S)$ be the system failure rate corres-

ponding to $\bar{F}(\bar{G})$. Since $r \geq s$, $\bar{F} \leq \bar{G}$. Using Theorem 1 of Esary and Proschan (1963) and the fact that $ph'(p)/h(p)$ is decreasing in p, we have:

$$\begin{aligned} R(t) &= r(t)\ \bar{F}(t)\ h'(\bar{F}(t))/h(\bar{F}(t)) \\ &\geq s(t)\ \bar{G}(t)\ h'(\bar{G}(t))/h(\bar{G}(t)) \\ &= S(t). \end{aligned}$$

If F and G are not absolutely continuous, consider instead the convolutions of F and G with an exponential distribution function with mean rate λ. Since these convolutions are absolutely continuous, the proof above applies. We then use the fact that TP_2 ordering of the resulting system reliability functions is preserved as $\lambda \to \infty$; since the convolution of F(G) and an exponential distribution with parameter λ converges in distribution to F(G) as $\lambda \to \infty$, we have the required result.||

Since $ph'(p)/h(p)$ is decreasing in p in the case of k-out-of-n systems of independent and identically distributed components, the following corollary is immediate.

Corollary 6.2.

Let h be the reliability function of a k-out-of-n system of independent and identically distributed components. Then $h(\bar{F}) \overset{TP_2}{<} h(\bar{G})$ whenever $\bar{F} \overset{TP_2}{<} \bar{G}$.

In the case of nonidentical components we show that if each of a set of nonidentical independent components is less (more) reliable than a component of type A, say, then a k-out-of-n system is less (more) reliable if the nonidentical components are used rather than a set of components of type A.

Theorem 6.3.

Let the reliability function of a k-out-of-n system of independent components be given by $h(\underline{p})$, where $\underline{p} = (p_1, \ldots, p_n)$ and p_i is the failure probability of component i.

(i) If $\bar{F}_i \overset{TP_2}{<} \bar{G}$, $i = 1, \ldots, n$ then $h(\bar{F}_1, \ldots, \bar{F}_n) \overset{TP_2}{<} h(\bar{G}, \ldots, \bar{G})$

(ii) If $\bar{F}_i \overset{TP_2}{>} \bar{G}$, $i = 1, \ldots, n$ then $h(\bar{F}_1, \ldots, \bar{F}_n) \overset{TP_2}{>} h(\bar{G}, \ldots, \bar{G})$

Proof: We prove only (i) since the proofs of (i) and (ii) are similar. Suppose first that $F_1, \ldots, F_n$ and G are absolutely continuous with failure rates $r_1, \ldots, r_n$ and s respectively, where $\bar{F}_i \overset{TP_2}{<} \bar{G}$ or, equivalently, $r_i \geq s$, $i = 1, \ldots, n$. Let $R(S)$ be the system failure rate corresponding to $F_1, \ldots, F_n (G, \ldots, G)$. Now $\bar{F}_i \leq \bar{G}$ for $i = 1, \ldots, n$ because $\bar{F}_i \overset{TP_2}{<} G$ for $i = 1, \ldots, n$. By Theorem 4 and (4.1) of Esary and Proschan (1963), we have:

$$
\begin{aligned}
S(t) &= \sum_{i=1}^{n} s(t) \quad \bar{G}(t) \left(\frac{\partial h/\partial pi}{h(\underline{p})} \right) \Bigg|_{\underline{p}=\underline{\bar{G}}(t)} \\
&\leq \sum_{i=1}^{n} r_i(t)\ \bar{F}_i(t) \left(\frac{\partial h/\partial pi}{h(\underline{p})} \right) \Bigg|_{p_j=\bar{F}_j(t),\ j=1,\ldots n} \\
&= R(t).
\end{aligned}
$$

As in the proof of the preceeding theorem, if any of $F_1, \ldots, F_n$, G are not absolutely continuous, we consider convolutions with the exponential distribution function and use an argument involving limits to obtain the required result. ||

The theorem above gives a way to provide upper and lower bounds on the reliability of a k-out-of-n system of nonidentical independent components. The general case is that in which one set of nonidentical independent components is "less reliable" than another set of nonidentical independent components. The question of interest is whether a system would be less reliable if the "less reliable" set of components were used rather than the other set. The answer is in the affirmative, provided that corresponding components bear to each other a relationship stronger than that involving comparison of reliability functions.

Theorem 6.4.

Let h be the reliability function of a system of n independent components. If $f_i \overset{TP_2}{<} g_i$ for $i = 1, \ldots, n$ then $h(\bar{F}_1, \ldots, \bar{F}_n) \overset{TP_2}{<} h(\bar{G}_1, \ldots, \bar{G}_n)$.

For the proof of Theorem 6.4 we need the following lemma.

Lemma 6.5.

Let h be the reliability function of a system of n independent components. If $f_1 \overset{TP_2}{<} g_1$ then $h(\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_n) \overset{TP_2}{<} h(\bar{G}_1, \bar{F}_2, \ldots, \bar{F}_n)$.

Proof of lemma.

Let $\{P_j\}$ denote the collection of minimal path sets of the system under consideration. For s and t satisfying $0 \le s < t$, let

$$\ell(\underline{x},\underline{y}) = \begin{cases} 1 & \text{if } \max_j \min_{i \in P_j} x_i \ge s \text{ and } \max_j \min_{i \in P_j} y_i \ge t \\ 0 & \text{otherwise} \end{cases}$$

If $x_2, \ldots, x_n$ and $y_2, \ldots, y_n$ are fixed then whenever $x_1 \le y_1$, we have

$$\begin{aligned} \ell^*(x_1, y_1) &\equiv \ell(x_1, x_2, \ldots, x_n;\ y_1, y_2, \ldots, y_n) \\ &\ge \ell(y_1, x_2, \ldots, x_n;\ x_1, y_2, \ldots, y_n) \\ &= \ell^*(y_1, x_1) \end{aligned}$$

To obtain this inequality, define the following functions for any $s \ge 0$:

$$\ell_s(x_1) = \begin{cases} 1 & \text{if } \max_j \min_{i \in P_j} x_i \ge s \\ 0 & \text{otherwise} \end{cases}$$

$$\ell_s(y_1) = \begin{cases} 1 & \text{if } \max_j \min_{i \in P_j} y_i \ge s \\ 0 & \text{otherwise} \end{cases}$$

Since $\ell^*(x_1,y_1) = \ell_s(x_1)\, \ell_t(y_1)$, we need to show that $\ell_s(x_1)\, \ell_t(y_1) \ge \ell_s(y_1)\, \ell_t(x_1)$. It suffices to show that $\ell_t(x_1)\ell_s(y_1) = 1$ implies that $\ell_s(x_1)\ell_t(y_1) = 1$: this follows immediately from the fact that $\ell_s(x)$ is increasing in x and decreasing in s.

Since $f \overset{TP_2}{<} g$ and $\ell^*(x_1, y_1) \geq \ell^*(y_1, x_1)$ for $x_1 \leq y_1$, we have, by Theorem 5.1 of Keilson and Sumita (1982) or by Proposition 8.4.2. of Ross (1983):

$$\ell^*(X_1, Y_1) \overset{st}{\geq} \ell^*(Y_1, X_1)$$

where $X_1(Y_1)$ has density $f_1(g_1)$ and X_1 and Y_1 are independent. Rewriting this in terms of ℓ, we have

$$\ell(X_1, x_2, \ldots, x_n; Y_1, y_2, \ldots, y_n) \overset{st}{\geq} \ell(Y_1, x_2, \ldots, x_n; X_1, y_2, \ldots, y_n)$$

Consequently:

$$E(\ell(X_1, x_2, \ldots, x_n; Y_1 y_2, \ldots, y_n)) \geq E(\ell(Y_1 x_2, \ldots, x_n; X_1, y_2, \ldots, y_n)).$$

Now let $X_2, \ldots, X_n, Y_2, \ldots, Y_n$ be independent random variables that are independent of X_1 and Y_1 also and suppose that X_i and Y_i have distribution function F_i for $i = 2, \ldots, n$. Then we have

$$E(\ell(X_1, X_2, \ldots, X_n; Y_1, Y_2, \ldots, Y_n)) \geq E(\ell(Y_1, X_2, \ldots, X_n; X_1, Y_2, \ldots, Y_n)).$$

Since $X_1, \ldots X_n, Y_1, \ldots, Y_n$ are independent, we use the definition of ℓ to rewrite this inequality as

$$h(\bar{F}_1(s), \bar{F}_2(s), \ldots, \bar{F}_n(s))\, h(\bar{G}_1(t), \bar{F}_2(t), \ldots, \bar{F}_n(t))$$

$$\geq h(\bar{G}_1(s), \bar{F}_2(s), \ldots, \bar{F}_n(s))\, h(\bar{F}_1(t), \bar{F}_2(t), \ldots, \bar{F}_n(t)).$$

This is true for any $s \leq t$.||

Proof of theorem: By repeated application of the lemma, we have

$$h(\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_n) \overset{TP_2}{<} h(\bar{G}_1, \bar{F}_2, \ldots, \bar{F}_n) \overset{TP_2}{<} h(\bar{G}_1, \bar{G}_2, \bar{F}_3, \ldots, \bar{F}_n) \overset{TP_2}{<} \ldots$$

$$\ldots \overset{TP_2}{<} h(\bar{G}_1, \ldots, \bar{G}_{n-1}, \bar{F}_n) \overset{TP_2}{<} h(\bar{G}_1, \ldots, \bar{G}_{n-1}, \bar{G}_n)$$

Since $\overset{TP_2}{<}$ is transitive, $h(\bar{F}_1, \ldots, \bar{F}_n) \overset{TP_2}{<} h(\bar{G}_1, \ldots, \bar{G}_n)$.||

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